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Properties of Caputo Operator and Its Applications to Linear Fractional Differential Equations

B. R. Sontakke, A. S. Shaikh
Department of Mathematics, Pratishthan Mahavidyalaya, Paithan - 431 107 Dist. Aurangabad, (M.S.) India
Department of Mathematics, Poona College of Arts, science & Commerce, Camp, Pune- 411001, (M.S.), India

Abstract
The purpose of this paper is to demonstrate the power of two mostly used definitions for fractional differentiation, namely, the Riemann-Liouville and Caputo fractional operator to solve some linear fractional-order differential equations. The emphasis is given to the most popular Caputo fractional operator which is more suitable for the study of differential equations of fractional order..Illustrative examples are included to demonstrate the procedure of solution of couple of fractional differential equations having Caputo operator using Laplace transformation. Itshows that the Laplace transforms is a powerful and efficient technique for obtaining analytic solution of linear fractional differential equations
Keywords: Fractional differential equations; The Riemann-Liouville and Caputo fractional derivatives, Laplace Transform method.

I. Introduction

Although the fractional calculus has a long history and has been applied in various fields in real life, the interest in the study of FDEs and their applications has attracted the attention of many researchers and scientific societies beginning only in the last three decades. Since the exact solutions of most of the FDEs cannot be found easily, the analytical and numerical methods must be used. During the last decades, several methods have been proposed to solve fractional partial differential equations, fractional integro-differential equations and dynamic systems containing fractional derivatives, such as Adomian decomposition method [1, 2, 3], He's variational iteration method [8], homotopy perturbation method [9], homotopy analysis method [5], existence and uniqueness results by using monotone method[4,6], Another powerful method which can also give explicit form for the solution is the Laplace transform method, which will allow us to transform fractional differential equations into algebraic equations and then by solving this algebraic equations, we can obtain the unknown function by using the Inverse Laplace Transform.

Fractional differential equations concerning the Riemann-Liouville fractional operators [7] or the Caputo derivative have been recommended by many authors. However, the Caputo (1967) definition of fractional derivatives not only provides initial conditions with clear physical interpretation but it is also bounded, meaning that the derivative of a constant is equal to 0.

In this paper, section 2 is begin by introducing some necessary definitions and properties of The Riemann-Liouville and Caputo fractional derivatives.In section 3 the Laplace transform and the inverse Laplace transform is discussed in details. In section 4, Illustrative examples are included to demonstrate the procedure of solution of fractional differential equation using Laplace transformation.

II. Definitions and Properties

2.1 Gamma function [12]

The Gamma function denoted by $\Gamma(z)$, is a generalization of factorial function$n!$ for complex argument with positive real part it is defined as

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt \qquad \text{ReZ} > 0$$

by analytic continuation the function is extended to whole complex plane except for the points {0, -1, -2, -3,…} where it has simple poles.

2.2 The Mittag-Leffler Functions [12]

While the Gamma function is a generalization of factorial function, the Mittag-Leffler function is a generalization of exponential function, first introduced as a one parameter function by the series

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \alpha > 0, \ \alpha \epsilon R, \ z \epsilon C$$

Later the two parameter generalization introduced by Agarwal

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta > 0, \quad \alpha, \beta \epsilon R$$

2.2 The Riemann-Liouville and Caputo Fractional Fractional Differential operator

(a) Suppose that $\alpha > 0, t > a, \alpha, \ a, \ t \ \epsilon \ R$. Then fractional operator

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(x)}{(t-x)^{\alpha+1-n}} \, dx; \qquad n-1 < \alpha < n$$

is called the Riemann-Liouville fractional derivative or Riemann-Liouville fractional differential operator of order α.

(b) Another definition that can be used to compute a differ integral was introduced by Caputo in the 1960s. The benefit of using the Caputo definition is that it not only allows for the consideration of easily interpreted initial conditions, but it is also bounded, meaning that the derivative of a constant is equal to 0. Suppose that $\alpha > 0, t > a, \alpha, \ a, \ t \ \epsilon \ R$. Then fractional operator

$$D_*{}^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(x)}{(t-x)^{\alpha+1-n}} \, dx \qquad n-1 < \alpha < n$$

is called the Caputo fractional derivative or Caputo fractional differential operator of order α [10]

2.3 Properties of Caputo Fractional Differential operator

- **Representation**

Let $n-1 < \alpha < n$, $n \in N$, $\alpha \in R$ and $f(t)$ be such that $D_*{}^\alpha f(t)$ exist. then

$$D_*{}^\alpha f(t) = I^{n-\alpha} D^n f(t)$$

This means that the Caputo fractional operator is equivalent to $(n-\alpha)$ fold integration after n^{th} order differentiation. While Riemann-Liouville fractional derivative is equivalent to the composition of same operator but in reverse order.

- **Interpolation**

Let $n-1 < \alpha < n$, $n \in N$, $\alpha \in R$ and $f(t)$ be such that $D_*{}^\alpha f(t)$ exist. Then following properties hold for the Caputo operator.

$$\lim_{\alpha \to n} D_*{}^\alpha f(t) = f^{(n)}(t),$$

$$\lim_{\alpha \to n-1} D_*{}^\alpha f(t) = f^{(n-1)}(t) - f^{(n-1)}(0)$$

- **Linearity:**

Let $n-1 < \alpha < n$, $n \in N$, $\alpha, \ \lambda \in C$ and functions $f(t)$and $g(t)$ be such that both $D^\alpha f(t)$ and $D_*{}^\alpha g(t)$ exist. the Caputo fractional derivative is a linear operator, i. e.

$$D_*{}^\alpha\big(\lambda f(t) + g(t)\big) = \lambda D_*{}^\alpha f(t) + D_*{}^\alpha g(t)$$

- **Non-commutation**

Let $n-1 < \alpha < n$, m, $n \in N$, $\alpha \in R$ and the function $f(t)$ is such that $D_*{}^\alpha f(t)$ exists.
Then in general

$$D_*{}^\alpha D^m f(t) = D_*{}^{\alpha+m} f(t) \neq D^m D_*{}^\alpha f(t)$$

- In general the two operators, Riemann- Liouville and Caputo, do not coincide, but if function $f(t)$ be such that $f^{(s)}(0) = 0$, $s = 0, 1, 2, \ldots n-1$ then the Riemann – Liouville and Caputo fractional derivatives coincides

$$D^\alpha f(t) = D_*{}^\alpha f(t)$$

- **Initial condition**

Consider the following differential equation

$$D^{\alpha} y(t) - y(t) = 0; \quad t > 0,\ n-1 < \alpha < n\ \in N, \text{(A)}$$

$$[\ D^{\alpha-k-1} y(t)]_{t=0} = b_k \qquad k = 0, 1, \quad 2, \ldots \quad n-1$$

$$D_*{}^{\alpha} y(t) - y(t) = 0; \quad t > 0,\ n-1 < \alpha < n\ \in N \quad \text{(B)}$$

$$y^{(k)}(0) = c_k k = 0, 1, \qquad 2, \ldots \quad n-1$$

Here it is worth to note that:

In (A) Riemann-Liouville fractional derivative is applicable and initial conditions with Fractional derivative are required. In such initial value problems solutions are practically useless, because there is no clear physical interpretation of this type of initial condition [12]. On the contrary in (B) the Caputo Fractional differentiation operator is applicable, standard initial conditions in terms of derivatives of integer order is involved. These initial conditions have clear physical interpretation as an initial position y(a) at point a, the initial velocityy'(a), initial acceleration y''(a)and so on.

This is a main advantage of Caputo operator over Riemann-Liouvillefractional operator.

- The Laplace transform of the Caputo Fractional derivative is a generalization of the Laplace

transform of integer order derivative, where n is replaced by $\alpha.$ The same does not hold for the Riemann-Liouville case. This Property is an important advantage of the Caputo operator over the Riemann-Liouville operator.

- Let $t > 0,\ \alpha \in R,\ n-1 < \alpha < n \in N$ then the following relation between the Riemann Liouville and the Caputo operator holds [11]

$$D_*{}^{\alpha} f(t) = D^{\alpha} f(t) - \sum_{k=0}^{n-1} \frac{t^{k-\alpha}}{\Gamma(k+1-\alpha)} f^{(k)}(0)$$

III. The Laplace Transform

3.1Definition:

Let f(t)be a function of a variable t such that the function $e^{-st} f(t)$is integrable in $[0, \infty)$for some domain of values of s. The Laplace transform of the function $f(t)$is defined for above domain values of s and it is denoted by [12] $L\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt.$

The Laplace transform of function $f(t) = t^{\alpha}$ is given forα as non-integer order $n-1 < \alpha \le n$

$$L\{t^{\alpha}\} = \frac{\Gamma(\alpha+1)}{S^{\alpha+1}}$$

3. 2Laplace Transform of the basic fractional operator

Suppose that p > 0, and F(s) is the Laplace transform of f(t) then following statements holds [12]

(a) The Laplace transform of Riemann-Liouville Fractional differential operator of order α is given by

$$L\{D^{\alpha} f(t)\} = s^{\alpha} F(s) - \sum_{k=0}^{n-1} s^{n-k-1} [D^k I^{n-\alpha} f(t)]_{t=0} \qquad n-1 < \alpha < n \qquad (3.1)$$

(b) The Laplace transform of Caputo Fractional differential operator of order α is given by

$$L\{D_*{}^{\alpha} f(t)\} = s^{\alpha} F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0), \qquad n-1 < \alpha \le n \in N \qquad (3.2)$$

Which can also be obtain in the form

$$L\{D_*{}^{\alpha} f(t)\} = \frac{s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)}{s^{n-\alpha}} \qquad (3.3)$$

(c) Let $\alpha,\ \beta,\ \lambda \in R,\ \alpha, \beta > 0,\ m \in N.$ Then the Laplace transform of the two – parameter function of Mittag-Leffler type is given by

$$L\{t^{\alpha m+\beta-1}E_{\alpha,\beta}{}^{(m)}(\pm\lambda t^{\alpha})\} = \frac{m!\, s^{\alpha-\beta}}{(s^{\alpha} \mp \lambda)^{m+1}}, \quad Re(s) > |\lambda|^{\frac{1}{\alpha}} \tag{3.4}$$

This formula is mainly used for solving FDEs.

Lemma1: Let $a \in R,\ \alpha \geq \beta > 0$, $s^{\alpha-\beta} > |a|$ we have following inverse Laplace transform formula

$$L^{-1}\left\{\frac{1}{(s^{\alpha} + as^{\beta})^{n+1}}\right\} = t^{\alpha(n+1)-1}\sum_{k=0}^{\infty}\frac{(-a)^{k}\binom{n+k}{k}}{\Gamma(k(\alpha-\beta)+(n+1)\alpha)}t^{k(\alpha-\beta)} \tag{3.5}$$

Proof :

$$\frac{1}{(s^{\alpha} + as^{\beta})^{n+1}} = \frac{1}{(s^{\alpha})^{n+1}\left(1 + \frac{a}{s^{\alpha-\beta}}\right)^{n+1}}$$

$$= \frac{1}{(s^{\alpha})^{n+1}}\sum_{k=0}^{\infty}\binom{n+k}{k}\left(\frac{-a}{s^{\alpha-\beta}}\right)^{k} \qquad since \quad \left[\frac{1}{(1+x)^{n+1}} = \sum_{k=0}^{\infty}\binom{n+k}{k}(-x)^{k}\right]$$

Applying inverse Laplace transform

$$= L^{-1}\left\{\frac{1}{(s^{\alpha})^{n+1}}\right\}\sum_{k=0}^{\infty}\binom{n+k}{k}(-a)^{k}L^{-1}\left\{\frac{1}{s^{k(\alpha-\beta)}}\right\}$$

$$= t^{\alpha(n+1)-1}\sum_{k=0}^{\infty}\frac{(-a)^{k}\binom{n+k}{k}}{\Gamma(k(\alpha-\beta)+(n+1)\alpha)}t^{k(\alpha-\beta)}$$

Lemma 2:$\alpha \geq \beta,\ \alpha > \gamma,\ a \in R,\ \ s^{\alpha-\beta} > |a|, \left|s^{\alpha} + as^{\beta}\right| > |b|$

$$L^{-1}\left\{\frac{s^{\gamma}}{(s^{\alpha} + as^{\beta} + b)}\right\} = t^{\alpha-\gamma-1}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-b)^{n}(-a)^{k}\binom{n+k}{k}}{\Gamma(k(\alpha-\beta)+(n+1)\alpha-\gamma)}t^{k(\alpha-\beta)+n\alpha} \tag{3.6}$$

Proof:

$$\left\{\frac{s^{\gamma}}{(s^{\alpha} + as^{\beta} + b)}\right\} = \frac{s^{\gamma}}{(s^{\alpha} + as^{\beta})(1 + \frac{b}{s^{\alpha}+as^{\beta}})} = \frac{s^{\gamma}}{(s^{\alpha} + as^{\beta})}\frac{1}{(1 + \frac{b}{s^{\alpha}+as^{\beta}})}$$

Since$\left|\frac{b}{s^{\alpha}+as^{\beta}}\right| < 1,\ and$ using $\left[\frac{1}{(1+x)} = \sum_{n=0}^{\infty}(-1)^{n}x^{n}\right]$

$$= \frac{s^{\gamma}}{(s^{\alpha} + as^{\beta})}\sum_{n=0}^{\infty}(-1)^{n}\left(\frac{b}{s^{\alpha} + as^{\beta}}\right)^{n} = \sum_{n=0}^{\infty}(-b)^{n}\frac{s^{\gamma}}{(s^{\alpha} + as^{\beta})^{n+1}}$$

Applying inverse Laplace transform and using Lemma 1, also$L^{-1}\{s^{\gamma}\} = \frac{t^{-\gamma-1}}{\Gamma(-\gamma)}$

$$= t^{\alpha-\gamma-1}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-b)^{n}(-a)^{k}\binom{n+k}{k}}{\Gamma(k(\alpha-\beta)+(n+1)\alpha-\gamma)}t^{k(\alpha-\beta)+n\alpha}$$

IV. Applications

In this section we obtain the solution of some linear fractional differential equations with Caputo operator using Laplace transform method.The Laplace transform method is one of the most powerful methods of solving LFDE switch constant coefficients. On the other hand it is useless for LFDEs with general variablecoefficients or for nonlinear FDEs.

Example 1. Consider the fractional differential equation of the form

$D_*{}^{\alpha} y(t) = f(t)$ with initial conditions

$y^{(k)}(0) = c_k; \quad k = 0, \quad 1, 2, \ldots \ n-1$

Where $D_*{}^{\alpha}$ denotes Caputo derivative and n is the smallest integer greater than α

Solution:

Suppose that $f(t)$ isa sufficiently good function i.e. Laplace transform of $f(t)$existApplying the Laplace transform on both the sides of equation we have,

$$\mathrm{L}\{D_*{}^{\alpha} y(t)\} = \mathrm{L}\{f(t)\}$$

Applying (3.2) on LHS we get, $s^{\alpha} Y(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} y^{(k)}(0) = F(s)$

Since Laplace transform is linear equation can be solved with respect to Y(s) as follows

$\mathrm{Y(s)} = \frac{F(s)+\sum_{k=0}^{n-1} s^{\alpha-k-1} y^{(k)}(0)}{s^{\alpha}}$ by using initial conditions

$\mathrm{Y(s)} = \frac{F(s)+\sum_{k=0}^{n-1} c_k s^{\alpha-k-1}}{s^{\alpha}}$

by Laplace transform of the two-parameter of Mittag-Leffler type as well as the linearity property it follows

$$Y(s) = \frac{F(s)}{s^{\alpha}} + \sum_{k=0}^{n-1} \frac{s^{\alpha-k-1}}{s^{\alpha}} c_k$$

Making use of (3.4),

$$= \frac{F(s)}{s^{\alpha}} + \sum_{k=0}^{n-1} L\{t^k E_{\alpha,k+1}(t^{\alpha})\} c_k = \frac{F(s)}{s^{\alpha}} + L\left\{\sum_{k=0}^{n-1} c_k t^k E_{\alpha,k+1}(t^{\alpha})\right\}$$

Then using the inverse Laplace transform y(t) can be found as

$$L^{-1}\{Y(s)\} = L^{-1}\left\{\frac{F(s)}{s^{\alpha}} + L\left\{\sum_{k=0}^{n-1} c_k t^k E_{\alpha,k+1}(t^{\alpha})\right\}\right]$$

$$y(t) = L^{-1}\left\{\frac{F(s)}{s^{\alpha}}\right\} + \sum_{k=0}^{n-1} c_k t^k E_{\alpha,k+1}(t^{\alpha}) = t^{\alpha-1} E_{\alpha,\alpha}(t^{\alpha}) * f(t) + \sum_{k=0}^{n-1} c_k t^k E_{\alpha,k+1}(t^{\alpha})$$

which isa required solution.

Example 2

Consider the Bagley- Torvik equation which arises in modeling the motion of a rigid plate immersed in a Newtonian fluid.

$$D_*{}^2 y(t) + 2D_*^{\frac{3}{2}} y(t) + 2y(t) = 8t^5; y(0) = 0, \quad y'(0) = 0$$

Solution:

Applying the Laplace transform on both the sides of above equation we have,

$$L\{D_*{}^2 y(t) + 2D_*^{\frac{3}{2}} y(t) + 2y(t)\} = L\{8t^5\}$$

$$s^2 Y(s) - sy(0) - y'(0) + 2\left[\frac{s^2 Y(s) - sy(0) - y'(0)}{s^{\frac{1}{2}}}\right] + 2Y(s) = 8\frac{5!}{s^6}$$

by initial conditions

$$s^2 Y(s) - s(0) - (0) + 2\left[\frac{s^2 Y(s) - s(0) - (0)}{s^{\frac{1}{2}}}\right] + 2Y(s) = 8\frac{5!}{s^6}$$

$$s^2 Y(s) + 2\frac{s^2 Y(s)}{s^{\frac{1}{2}}} + 2Y(s) = 8\frac{5!}{s^6}$$

$$\mathrm{Y(s)}\left[s^2 + 2s^{\frac{3}{2}} + 2\right] = 8\frac{5!}{s^6} = Y(s) = \frac{8\times 5!}{s^6\left(s^2+2s^{\frac{3}{2}}+2\right)}$$

Applying inverse Laplace transform both side of above equation

$$L^{-1}\{Y(s)\} = L^{-1}\left\{\frac{8\times 5!}{s^6\left(s^2+2s^{\frac{3}{2}}+2\right)}\right\} = L^{-1}\left\{\frac{8\times 5!\ s^{-6}}{\left(s^2+2s^{\frac{3}{2}}+2\right)}\right\}$$

Applying (3.6) $\alpha = 2,\ \beta = \frac{3}{2},\ \gamma = -6$

$$y(t) = L\left\{\frac{8\times 5!\ s^{-6}}{\left(s^2+2s^{\frac{3}{2}}+2\right)}\right\} = 8\times 5!\, t^6 \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-2)^{n+k}\binom{n+k}{k}}{\Gamma(\frac{1}{2}k+2(n+1)+6)} t^{k\frac{1}{2}+2n}$$

This is a required solution.

V. Conclusions

The fractional derivatives are described in the Caputo sense, obtained by Riemann-Liouville fractional integral operator. InCaputo Fractional differential equation initial conditions have clear physical interpretation which is a main advantage of Caputo operator over Riemann-Liouville fractional operator. Solving some problems show that the Laplace transform is a powerful and efficient technique for obtaining analytic solution of linear ordinary fractional differential equations.

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